Arithmetic from Kant to Frege: Numbers, Pure Units, and the Limits of Conceptual Representation

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There is evidence in Kant of the idea that concepts of particular numbers, such as the number 5, are derived from the representation of units, and in particular pure units, that is, units that are qualitatively indistinguishable. Frege, in contrast, rejects any attempt to derive concepts of number from the representation of units. In the *Foundations of Arithmetic*, he softens up his reader for his groundbreaking and unintuitive analysis of number by attacking alternative views, and he devotes the majority of this attack to the units view, with particular attention to pure units.[1] Since Frege, the units view has been all but abandoned. Nevertheless, the idea that concepts of number are derived from the representation of units has a long history, beginning with the ancient Greeks, and was prevalent among Frege's contemporaries. I am not interested in resurrecting the units view or in righting wrongs in Frege's criticisms of his contemporaries. Rather, I am interested in the program of deriving concepts of number from pure units and its history from Kant to Frege. An examination of that history helps us understand the units view in a way that Frege's criticisms do not, and in the process uncovers important features of both Kant's and Frege's views. I will argue that, although they had deep differences, Kant and Frege share assumptions about what such a view would require and about the limits of conceptual representation. I will also argue that they would have rejected the accounts given by some of Frege's contemporaries for the same reasons. Despite these agreements, however, there is evidence that Kant thinks that space and time play a role in

[1] Frege (1884), English translation (1950). Frege commits a total of 27 sections at the beginning of the *Foundations* to attacking his contemporaries on the concept of number. He begins with his crucial arguments that number is neither a property of external things nor something subjective, a theme that reappears throughout his polemic. He then devotes the next 16 sections to attacking the idea that numbers could be derived from units (10 sections attack pure units in particular, and two are specifically directed against an appeal to space and time).

doi:10.1017/S1358246108000088

overcoming the limitations of conceptual representation, while Frege argues that they do not.

Because this paper examines views from Kant to Frege, in other respects it must remain fairly focused. For example, it concentrates on accounts of number concepts themselves, and only indirectly considers accounts of arithmetical operations. Furthermore, Kant thought that temporal succession and the representation of order are central to arithmetical cognition, but I will highlight the role of space and the representation of collections. This paper is an exploration of one theme found in Kant's statements about arithmetical cognition; a complete account of Kant's views will substantially qualify the view I attribute to him here.

This paper will begin with some background concerning Kant's philosophy of mathematics and Greek mathematics before turning to Frege's contemporaries and how Kant would have evaluated them. It will then take up Frege's criticisms of his contemporaries and the relation between Frege's and Kant's views. Finally, it will consider what each has to say about the possibility of a role for space or time.

1. Kant and concepts of number from pure units

Kant's philosophy of mathematics rests on his theory of magnitudes. Before the nineteenth-century arithmetization of mathematics placed natural numbers at its foundation, it was common to describe mathematics as the science of magnitudes, a view that can be traced to the Greek mathematical tradition, in particular to the Eudoxian theory of proportions found in Euclid's Books V and VII.[2] The Eudoxian theory provides a general treatment of magnitudes that does not presuppose numbers and yet covers both continuous magnitudes such as lines, planes, and volumes, and discrete magnitudes, such as numbers.[3] Magnitudes are homogeneous within their kinds; that is, magnitudes

[2] Friedman (1992a), pp. 110–3, points out the connection between Kant's theory of magnitudes and the Eudoxian theory. I develop a broader interpretation of Kant's theory of magnitudes and its relation to the Eudoxian theory in Sutherland (2004).

[3] Although the theory treats both continuous and discrete magnitudes, the treatment of the continuous and the discrete are split between Books V and VII respectively. Algebra traces its origin through an arithmetical tradition leading back to Diophantus, but the development of algebra and its application to geometry in the early modern period led to its being viewed as a universal mathematics of all magnitudes, corresponding to the

that are homogeneous can be composed to constitute more of exactly the same kind—lines with lines, areas with areas, and so on.

Kant follows the Greek conception when he defines a magnitude as a homogeneous manifold, but he moves beyond it to provide a deeper analysis of homogeneity and a theory of our cognition of homogeneous manifolds. In Kant's view a homogeneous manifold exhibits numerical difference without qualitative difference. In other words, a homogenous manifold consists of a manifold of parts that are qualitatively identical.[4] This fact about homogeneous manifolds has important consequences for our cognition of them. Kant states in the Amphiboly of the *Critique of Pure Reason* that Leibniz's Principle of the Identity of Indiscernibles (PII) could not be disputed if objects were cognized solely by means of the understanding, and hence by means of concepts alone, without the aid of intuition (A263–4/B319–20).[5] This important counterfactual asserts a fundamental limitation of conceptual representation—our inability to represent numerical diversity with qualitative identity by means of concepts alone. Kant emphasizes this limitation by asserting that an attempt to use concepts alone to represent a multiplicity of identical things inevitably collapses into the representation of just one thing.[6]

Eudoxian theory that bridges both continuous and discrete magnitudes. For a fuller discussion of this point, see Sutherland (2006).

[4] For a defense of the claim that Kant's analysis of homogeneity is an extension of the Greek notion of homogeneity, see Sutherland (2004).

[5] Quotations from the *Critique of Pure Reason* closely follow, with occasional modifications, the Paul Guyer and Allen Wood (1997) translation. All other references to Kant's work will be to volume and page number, separated by a colon, of the Akademie edition of *Kants gesammelte Schriften*.

[6] Kant states:

> According to mere concepts of the understanding, it is a contradiction to think of two things outside of each other that are nevertheless fully identical in respect of all their inner determinations (of quality and quantity); it is always one and the same thing thought twice (numerically one) (Ak 20:280).

A few pages later Kant adds that conceptual representation alone would 'bring the whole of infinite space into a cubic inch and less ...' (20:282). This work is translated as 'What Real Progress Has Metaphysics Made in Germany since the Time of Leibniz and Wolff?', in H. Allison and P. Heath (eds.), (2002) *Theoretical Philosophy after 1781*, The Cambridge Edition of the Works of Immanuel Kant (Cambridge: Cambridge University Press), 337–424.

Consequently, concepts on their own cannot represent a homogeneous manifold.

In contrast, Kant denies that PII is true of appearances, because they are objects of intuition, and intuition allows us to represent numerical difference with qualitative identity.[7] Since intuition can represent numerical difference without qualitative difference, it allows us to represent a homogeneous manifold, which I have argued elsewhere is in Kant's view a fundamental role for intuition in mathematical cognition.[8]

It is more difficult to see what role intuition should play in Kant's philosophy of arithmetic than in his geometry. The representation of succession is undoubtedly fundamental, but the role of intuition in representing discrete magnitudes is also central. When Kant discusses arithmetical cognition in the *Critique of Pure Reason*, he refers to discrete magnitudes such as fingers or dots on a page, which are both discrete and spatially disconnected (B15-6). They are related to our representation of units in some way similar to the way that a drawn triangle is related to our a priori construction of a triangle.[9]

Kant's treatment of discrete magnitudes suggests another important connection between Kant and Greek mathematics. Reconstructing the

[7] Kant makes this clear in the Amphiboly, which emphasizes this feature of space and at the same time its composability:

> (. . .) multiplicity and numerical difference are already given us by space itself as the condition of outer appearances. For a part of space, although completely similar and equal to another part, is nevertheless outside of it, and is for that very reason a different part from that which abuts it to constitute a larger space (A264/B320).

He reaffirms this later in the Amphiboly, stating:

> The concept of a cubic foot of space, wherever and however often I think it, is in itself always completely the same. Yet two [distinct] cubic feet of space are nevertheless distinguished in space merely through their locations (*numero diversa*) (A282/B338).

[8] See Sutherland (2004). Geometry provides the paradigm for the role of intuition in the representation of a homogeneous manifold. To quickly summarize the broader view: the categories of quantity—unity, plurality, and allness—are used to cognize the part-whole relations of continuous regions in space, which provides a mereological basis for our cognition of the composition relations among the parts of space. (It is worth noting that the role of intuition in allowing us to represent a homogeneous manifold has been overlooked, but it is not the only role for intuition in mathematical cognition).

[9] I will come back to this relation below.

Greek conception of number is a formidable challenge; I will limit myself to highlighting a few relevant features.[10]

The Greek term for number, '*arithmos*', had different senses. The thinnest sense seems to have been nothing more than a particular *collection* of things, such as the number of sheep in a particular field or the number of shoes in my closet, in roughly the way we think of a particular set. This notion of *arithmos* presupposes a choice of unit, such as a shoe or pairs of shoes. Another closely related sense of *arithmos* was the number that resulted from enumerating the members of a collection, which is sometimes called the 'counting-number' of the counted collection. It, too, presupposes the choice of a unit, which is often called a 'counting-concept.'[11] There is also a further conception of number apart from a kind of thing, one that is based on collections of 'pure' units. The motive for pure units may arise from the desire for a general representation of number apart from any kind of thing counted.[12] The close connection of *arithmos* to particular collections of things and the notion of number as a collection of units were influential into the nineteenth century.

We can distinguish two sorts of purity that are included in the generality to which pure units aspire. Consider the seven samurai. In the first sort of purity, we would like the units to represent any seven samurai, not the seven masterless samurai who defended a small village in the sixteenth century. This requires that the units of the collection are 'equal and not in the least different from each other,' as Plato puts it.[13] The units could still have the characteristics common

[10] My comments rely primarily on Jakob Klein (1968), especially Chapter 6; §1 of Stein (1990); and William Tait (2005), especially §9.

[11] Interpretation is difficult, but *arithmos* in this sense might conceivably belong *only* to the particular collection counted, so that the eight of these sheep would be distinct from the eight of those sheep. Rogers Albritton first suggested to me that considering numbers as abstract particulars is at the very least an interpretative possibility that requires consideration. Klein seems at least to make room for this reading; see Klein (1968), 46–7. However, *arithmos* as counting number might be thought of as a species under which collections of a particular size fall, so that the eight of these sheep is the same eight as the eight of those sheep. On this view, particular number words, such as 'eight' can be viewed as a common name of collections of a particular size. See Stein (1990), 163–4, and Tait (2005), 238–9.

[12] The generality of such a representation might be thought a necessary presupposition of the enumeration that results in counting-numbers, and hence it might be thought to be more fundamental. See Klein (1968), 49.

[13] Plato's *Republic*, trans. by G. Grube, (Indianapolis: Hackett, 1974, revised by C.D.C. Reeve in 1992), 526 A; quoted in Klein (1968), 24.

to all the things counted (that is, belonging to the Japanese warrior class) but the units would not be distinguished from each other by any further characteristics (such as being old, or short, or hot-headed). In the second sort of purity, we would like to represent seven of anything, be they seven samurai, seven swans-a-swimming, or the seven habits of highly effective people. This requires that the units also have no characteristics common to any particular *kind* of thing, such as being a samurai. Of course, if a pure unit has no characteristics at all other than being a unit, then both kinds of generality would be achieved, but the first kind of generality raises particular challenges for a conception of number based on units. How can we represent the units of a collection as distinct from each other without representing them with distinguishing characteristics? If we do not represent any distinguishing characteristics, then we are only left with the representation of a samurai, for example, with nothing to distinguish seven of them. But if we represent each samurai with distinguishing properties, then we lose the generality of the representation, and it will only represent a collection of samurai with those particular distinguishing characteristics. There is, then, a *prima facie* tension between a completely general representation of number by means of units and the particular characteristics required to distinguish those units. I will call this the 'pure plurality problem'.

The idea that number is based on pure units bears a striking resemblance to what I have said about Kant's appeal to a homogeneous manifold of discrete units. In Kant's view, concepts on their own could never represent a collection of units as distinct and yet without distinguishing characteristics, since concepts on their own represent by means of qualities, which are distinguishing characteristics. Intuition, on the other hand, allows us to represent pure numerical difference without specific difference, which is just what is required for the representation of pure units.

Before saying more, I would like to lay to rest a possible confusion. First, the units I have described would be pure in the sense of being completely free of qualitative differences. This is not to be confused with Kant's notion of purity in relation to the a priori, which requires that there be no admixture of anything empirical. Any sense of purity associated with the a priori is in addition to purity in the sense of being qualitatively identical, and unless I explicitly indicate otherwise, I will mean the latter. For all I have said, someone might think that we could in some way rely on empirical experience, perhaps in combination with abstraction, to represent pure units. Second, Kant indeed holds that arithmetic is a priori and hence not dependent on particular empirical experience for its justification, even if we use empirical intuitions such as fingers as an aid. He holds that, despite the aid of empirical intuition in their representation, number concepts themselves are pure in the sense of a priori. In that case, however, what is it that the intuition of the fingers aids us in representing when 'seeing the number 12 arise' (B16)? It is plausible that in Kant's account, at least part of the answer is the pure intuition of different locations in space underlying the empirical intuition of the fingers. The resulting representation of units would be pure in both senses—qualitatively identical and free of empirical content.

What textual evidence do we have that Kant thinks that arithmetical cognition in particular rests on the representation of pure—that is, qualitatively identical—units? In addition to referring to fingers and dots on a page in his discussion of the role of intuition in arithmetical cognition (B16), Kant also refers to strokes and beads of an abacus:

> (. . .) the concept of magnitude seeks its standing and sense in number, but seeks this in turn in the fingers, in the beads of an abacus, or in strokes and points that are placed before the eyes. The concept is always generated a priori, together with the synthetic principles or formulas from such concepts (. . .) (A240/B299).

Kant states explicitly that these empirical representations aid us in forming an a priori representation of the synthesis of units. It would be natural to suppose that an a priori and fully general representation of numbers based on units would have no admixture of the empirical in it, and hence would retain none of the distinguishing empirical characteristics of fingers or dots or beads or strokes, representing instead qualitatively identical units.

Comparison to Kant's comments on the relation between a drawn triangle and the a priori representation of a triangle is helpful on this point. Kant states that

> The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle (A714/B742).

A similar attention to construction in the case of arithmetic and hence a similar abstraction from the 'entirely indifferent' properties of units would lead to a representation of units as qualitatively identical. In other words, our fingers would be an aid to the a priori representation of units distinguished only by their difference in spatial location.[14]

Kant's descriptions of arithmetical cognition as cognition of magnitudes provide more direct evidence of this view. Any reference to magnitudes is a reference to a homogeneous manifold in intuition, and as I mentioned above, in Kant's view a homogeneous manifold consists of qualitatively identical elements. Moreover, Kant repeatedly describes arithmetical cognition as a synthesis of *homogeneous* units in particular. In the Schematism of the *Critique of Pure Reason*, for example, Kant states that 'the pure schema of magnitude (*quantitas*) is a representation that summarizes the successive addition of one to one (homogeneous)' (A142–3/B182; see also A164/B205, A242/B300).

In Kant's time, however, there was also another, less stringent notion of homogeneity that was often mentioned as a condition of counting. Counting was said to require that one employs a counting-concept under which all the things to be counted fall; to count a collection of jaguar and tapir, for example, one would have to ascend at least to the concept of mammal, say, and count the items under this concept. As mentioned above, the counting-concept requirement is found in ancient Greek arithmetic, but the role of counting-concepts should not be restricted to acts of counting; any representation of a collection was thought to require a common concept under which the elements fall. In Kant's time, the things that fall under such a concept were said to be homogeneous with respect to that concept. Clearly, this sort of homogeneity does not require that the units be qualitatively identical.[15]

I believe, however, that it is precisely here that Kant, following his deeper analysis of the Greek notion of homogeneity, introduces a *further* notion of homogeneity whose representation is required for a priori knowledge in geometry and arithmetic. This homogeneity is the property of numerical difference without specific difference, a property of magnitudes that intuition allows us to represent. The fact that intuition allows us to represent this further kind of homogeneity constitutes a fundamental role for intuition in all mathematical cognition.[16] This account of homogeneity derives further support from, on the one hand, the conception of early modern

[14] I say more below about only attending to the action of construction in my discussion of schemata below.

[15] Interpreting Kant's notion of homogeneity along these lines is prominent in various interpretations, in particular Longuenesse (1998).

[16] For a more detailed defense of this view, see Sutherland (2004).

algebra as a science of magnitude common to geometry and arithmetic, and on the other, the connection between early modern algebra and the Eudoxian theory of proportions.[17] Kant's deeper notion of homogeneity does not displace the common-concept sense of homogeneity; the representation of a collection of seven samurai still requires the common concept of samurai. Nevertheless, a full account of our representation of seven samurai as a collection of seven also requires the deeper notion of homogeneity.[18]

There is another interpretive possibility that seems to obviate the representation of pure units. Kant distinguishes between images and schemata. As an example of an image of the number five, Kant refers to five points set alongside one another. The schema, in contrast, allows us to think of a number in general and is a universal procedure of imagination for providing an image for a number concept. In the case of arithmetic, Kant states

> (. . .) the pure *schema* of magnitude (*quantitas*), as a concept of the understanding, is *number*, a representation which summarizes the successive addition of one to one (homogeneous). Thus number is nothing other than the unity of the synthesis of the manifold of a homogeneous intuition in general, because I generate time itself in the apprehension of the intuition (A142–3/B182).

These passages can leave the impression that arithmetical cognition is based only on a rule of successive synthesis in time, and hence suggest that the representation of pure units in space, which might seem to fall on the side of images, are strictly speaking unnecessary. Indeed, these passages may be taken as evidence of a rather abstract view of

[17] For more on this additional support, see Sutherland (2006).

[18] There are two cases to consider. First, if we call to mind a general representation of any seven samurai, then we do not want to represent the individual samurai as possessing particular distinguishing properties, and the deeper homogeneity of intuition allows us to accomplish this. Second, if we have seven particular samurai before us and we wish to represent them as seven, the homogeneity of intuition may allow us to abstract from their distinguishing properties, in the way that we can appeal to our fingers in the representation of $5 + 7 = 12$. In both cases, however, the account may be more complicated; it may be that once we have attained pure a priori concepts of numbers and pure a priori cognition of arithmetical truths such as $5 + 7 = 12$, we can directly apply those concepts (and truths) to particular collections. Either way, however, there is a further issue that I will address below: Kant may think that the role of the homogeneity of intuition is to allow the concepts of number to arise or to establish their objective validity in a way that does not undermine their generality.

arithmetic as mere rules or techniques of calculation for finding the magnitudes of objects.[19] Most importantly, however, is the point Kant makes explicit in the geometrical case: the schema of a triangle is a rule of the synthesis of imagination for the construction of pure figures in space, and its being a rule rather than an image accounts for the generality of the conclusions we base on geometrical constructions (A140–1/B179–80). There are important differences between the generality of geometrical claims and arithmetical claims such as $5 + 7 = 12$, as Kant himself emphasized. Yet, if Kant appeals to schemata to explain the generality of geometrical claims, it seems possible that he also appeals to schemata to explain any generality underlying particular arithmetical claims or particular numerical concepts. In short, if the generality of mathematical claims rests on schemata (understood as rules for construction) rather than images, then it seems there is no need to represent pure units to achieve a fully general representation of number.[20] If successful, Kant's appeal to rules to explain generality would circumvent the pure plurality problem that arises from an attempt to explain the generality of our representation of five, for example, through the representation of pure units.

This shows that Kant's view is more sophisticated and complex than the simple pure units view outlined above. While Kant does appeal to schema to explain the generality of mathematical claims, the representation of pure units still plays an essential role in our cognition of fundamental a priori arithmetical truths, such as $5 + 7 = 12$. At the very least, Kant seems to think that the representation of pure units is required for us to acquire the concepts of number (to 'see the number 12 arise') or to establish the objective validity of our basic arithmetical concepts, and to do so in a manner that preserves the generality achieved by appealing to schemata. In short, what Kant claims in the Schematism is compatible with the further claim that when we take things like fingers as an aid in our pure a priori cognition of $5 + 7 = 12$, our taking them as representations of pure units is part of the schema of number itself. In fact, this last possibility is suggested by the Schematism passage just quoted, which refers to the successive addition of '(one to one) homogeneous', that is, homogeneous units.

[19] Friedman (1992a), pp. 112–22, develops this line of thought with some care. For important modifications to this view, see Friedman (2000).

[20] Friedman (1992a), pp. 122–29, identifies the schemata for geometrical constructions with Euclidean constructions, and points out that in Kant's view, the problem of deriving general conclusions from particular geometrical figures is solved by an appeal to schemata rather than images (p. 90).

I have given textual and interpretive grounds for thinking that in Kant's view, arithmetical cognition requires the representation of pure units. As with so many of Kant's views, one could wish for clearer expressions of it in the *Critique*. Kant comes close to making the qualitative identity of units explicit in the Discipline of Pure Reason, where he states:

> The general synthesis of *one and the same* [von einem und demselben] in time and space, and the magnitude of an intuition in general (number) that arises there from, is a business of reason through the construction of concepts, and is called mathematical (A724/B752, my emphasis).

On my interpretation, the explicit mention of multiple instantiations of 'one and the same in time and space' is a reference to units distinguished only by their location in time and space and to Kant's discussion of numerically distinct indiscernibles in the Amphiboly.

The clearest statement of the pure units view is not found in Kant himself, but in his foremost defender and expositor Johann Schultz. His *Prüfung der Kantischen Critik der reinen Vernunft*, published in 1789, is an examination and defense of Kant's philosophy with a particular emphasis on his philosophy of mathematics. Schultz's work and the correspondence between them are valuable resources for insights into Kant's views.[21]

As is well known, Schultz disagrees with Kant in thinking arithmetic is grounded in axioms, in particular, the axioms of commutivity and associativity.[22] Schultz states that the fact that mathematics

[21] Schultz had been Kant's student and later wrote a review of Kant's *Inaugural Dissertation* that influenced the development of Kant's thought. Encouraged by Kant, he published a summary and explanation of Kant's views (Schultz, 1784). Kant thought well enough of it that at one point he planned to use it as a textbook for his metaphysics course. Schultz (1789) is less of a summary and includes more philosophical analysis. (Here below, all references to Schultz are to this latter work with page numbers in brackets). In addition, Schultz wrote at least seven reviews supporting Kant's philosophy.

Schultz lived in Königsberg and became its court chaplain; he also received a professorship in mathematics at the university, most likely with Kant's support. The two would have had opportunity to converse, but they nevertheless communicated at least in part through letters; we have at least a dozen from Kant to Schultz. See Kuehn (2001), and James C. Morrison introduction to the English translation of Schultz (1784).

[22] Schultz (1789), 221–3, also identifies two postulates of arithmetic, both of which concern magnitudes and emphasize homogeneity. In

rests on intuition can be shown both from the nature of its axioms and postulates and from the concept of counting itself. We can set aside Schultz's views on the axioms and postulates, for it is what he says about the concept of counting that is of particular interest to us. Schultz states:

> When I am to add 5 to 7, for example, I must first individually represent to myself the units of the number 5, and hence represent $1 + 1 + 1 + 1 + 1$. Now every one of these units, the *quantity as well as the quality*, thus in itself, is completely identical [ganz einerlei], and thus the understanding cannot differentiate them by the smallest *inner* characteristic, consequently it cannot differentiate them through concepts. It is therefore only possible through *outer* or *sensible* characteristics, that is, through those that do not consist in a concept but an *intuition* (...) Things that in respect of their inner characteristics are perfectly identical [vollkommen einerlei], and hence completely indistinguishable for the mere understanding, we cannot distinguish in any other way than by representing them either *at different places of space* or *in different points in time* (225–6).

This passage reflects Kant's own claims about the limits of conceptual representation and provide strong indirect evidence that Kant thought that our representation of numbers depends in some way on the representation of pure units.[23]

More would be required to defend and elaborate this interpretation of Kant. In particular, one would need to fill out the account of the Schematism and explain the relation between the representations of temporal succession and of pure units, but I will not attempt to do so here. Instead, I would like to concentrate on the idea that our concepts of number depend on the representation of units—its historical influence and its fate—from Kant to Frege. When discussing Kant in what follows, I will focus on his views on the limitations on conceptually

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keeping with my interpretation Schultz is attempting to articulate the way in which Kant's philosophy of arithmetic rests on Kant's more general theory of magnitudes.

[23] Schultz also echoes Kant when he describes the consequences of attempting to represent pure units without appealing to intuition:

> The help of *intuition* is required, without which the concept of their *plurality* would be wholly impossible; rather, the understanding would have to think them all together as absolutely [schlechterdings] *eadem numero*, that is, as *only one* (226).

representing pure units and the role of intuition in overcoming those limitations.

2. Numbers from units in Frege's contemporaries

Those of Frege's contemporaries who appealled to units were inclined to empiricism and, in so far as they considered the basis of mathematical cognition, leaned toward empirical psychology. Several espoused versions of formalism. In general, they do not appear to be particularly interested in Kant. In considering how Kant would react to them, I will set aside his most obvious criticism of empiricism: that no account that relies on empirical experience for justification of mathematical claims can explain the a priori status of mathematics. Likewise, in considering Frege's criticisms I will set aside arguments against the psychologism and subjectivism of mental abstraction as well as his arguments against formalism. I will instead focus on the way in which Frege's contemporaries thought that concepts of number were derived from the representation of units, and how Kant and Frege would or did respond to that view in particular.

Before I begin, a few points about terminology will be helpful. First, the German term '*Anzahl*' was used in rough correspondence to the Greek notion of *arithmoi*, that is for a collection or a counting-number of a particular collection. The German term '*Zahl*' was also used in this way, but '*Zahl*' was first and foremost used for the notion of an individual number, such as the number 5, apart from any collection; some who wrote on mathematical foundations reserved it for the latter. Usage was by no means settled, however, and these terms were used in different ways, sometimes even interchangeably.[24] To help avoid confusion, I will use 'number' only to translate '*Zahl*' and will leave '*Anzahl*' untranslated.

Second, in its most general use the German term '*gleich*' was and remains highly ambiguous. It is often translated 'equal,' though it is also often translated 'identical', which is itself ambiguous. In many contexts, it is probably best translated by the correspondingly ambiguous term 'same'. When describing a relation between two or

[24] I am setting aside debates about various connotations that have been attributed to *Anzahl* and *Zahl* by philologists; the distinction I describe is paramount. For more details on these disputes, see Stosch (1772); Adelung (1793); Eberhard (1819), among others.

more things, '*gleich*' has at least five distinct senses, three of which correlate with three senses of identity:

1. Similar, that is, the same with respect to some property, relation, or circumstance; e.g. the same in color;
2. Equal, that is, the same with respect to particular quantitative properties; e.g. the same in length or volume;
3. Practically Identical, that is, the same with respect to all readily noticeable intrinsic properties; e.g. the sameness of some identical twins;
4. Qualitatively Identical, that is, completely the same with regard to all intrinsic properties; e.g. the sameness of two identical rain drops (supposing that they in fact share all intrinsic properties);
5. Strictly or Numerically Identical, that is, one and the same; e.g. the sameness of Cicero and Tully.

Only the fifth sense of equal and identical are absolute; the first four senses are all notions of equal or identical in some respect, that is, in respect to some property or kind of property.[25] One must keep these possible senses of equality and identity in mind when reading Frege's contemporaries.

There is, however, a further use of '*gleich*' that is crucial to take into account:

6. Equipollent, that is, the elements of one collection stand in a one–to–one correspondence with the elements of another collection.

This is also a notion of 'equality in some respect,' but this sort of equality asserts a one–to–one correspondence between the elements of one collection and the elements of another without further conditions on the elements themselves. This sense of equality *between collections* must be sharply distinguished from any requirement of sameness *among elements of a collection*. '*Gleich*' is used for both. Moreover, it would be easy to run them together, since the relation of equipollence and the representation of pure units might both be thought to play a role in the representation of the natural numbers. As we will see in the next section, failing to carefully distinguish them can lead to misunderstandings.

With these preliminaries aside, let us turn to Frege's contemporaries. Rudolph Lipschitz opens his 1877 *Lehrbuch der Analysis* with

[25] I would like to thank William Tait for prompting me to make this last point explicit.

a discussion of number comprising just two short paragraphs.[26] He states that the concept of *Anzahl* arises from regarding separated things and neglecting their characteristics, adding that one who wishes to obtain an overview of certain given things will begin with a certain thing and add a new thing to the earlier ones. This procedure is called counting and results in a fully determinate concept of number [*Zahl*] for a particular collection of given things.

Lipschitz is so brief that it is hard to be sure of his position. In fairness, it should be noted that some of those whom Frege attacks, and in particular Lipschitz, do not show great interest in the foundations of arithmetic, and seem eager to get on to analysis, analytic functions, or other topics. Regardless, Lipschitz does not acknowledge the need for generality. From the Kantian perspective, Lipschitz simply fails to see what would be required of a general concept of number based on units and hence fails to address the pure plurality problem.[27]

I turn next to Ernst Schröder. His 1873 *Lehrbuch der Arithmetik und Algebra* sets out to give an account of the natural numbers, or *natürlichen Zahlen*, but immediately begins with the conditions of counting, and instead discusses *Anzahl*, where that is understood as a number of things.[28]

Schröder calls anything that is to be counted a unit [*Einheit*] (5), and says that the conditions of counting require that units be distinguished from each other in our representation of them through some characteristic (3). Counting also requires that the objects appear to us to be similar [*ähnlich*] to each other, though this similarity condition is extremely minimal, for it can be satisfied by any accidental relation, such as being counted by us. Schröder slides from similarity [*ähnlich*] to equality [*gleich*] when he states that the question of the *Anzahl* of things arises only when the things are viewed as equal to each other; clearly, he has in mind the minimal understanding of '*gleich*' as similarity, corresponding to sense 1 listed above.

Schröder distinguishes between the equality among units of a collection and equality between collections. The latter is defined as an equipollence between two collections, that is, two collections are said to occur 'in equal *Anzahl* [*in gleicher Anzahl*]' when there

[26] Lipschitz (1877).

[27] It is possible that Lipschitz would simply reject the pure plurality problem; his views might approach those of Mill in this respect. Unfortunately, a discussion of Mill is beyond the scope of this paper.

[28] Schröder (1873). All references to Schröder will be to this work (page numbers in brackets).

is a 1–1 correspondence between their members (8). This, however, imposes no further conditions on the elements themselves.

Since Schröder's account of the notion of equality among units is extremely minimal and does not rule out distinguishing characteristics, his units are not pure. From a Kantian perspective, Schröder's treatment of *Anzahl* fails to acknowledge the need for a general representation of units.

This is only half of Schröder's account, however, for he has a different characterization of number itself, that is, *Zahl*. (It is noteworthy that Frege does not refer to this second characterization in his criticisms). Schröder states that number arises by means of drawing a stroke [*Strich*] corresponding to each element in a collection. He calls the strokes 'ones' or 'oners,' which are represented in his text by the numeral '1'.[29] He states that a natural number is a sum of oners, where by 'sum' he appears to simply mean a collection.[30] Thus, a number is a special case of a collection in which the units are oners, which are used to illustrate a particular collection of units in respect of their plurality.

Oners, like any other collection of units, can also stand in relations of equipollency. The convenience of numbers in determining equipollency relations among collections recommends them and leads to their use as a measure of frequency or plurality.

After saying that number is an illustration of units in respect of their frequency, Schröder states that with collections of oners, the very least of things is illustrated—'so little, that if the one is left out, nothing at all would remain'. He concludes that 'the abstraction in the illustration–process of counting is the greatest,' and entails the greatest simplification of representations (6). While numbers are identified with the collections of oners themselves, the maximal abstraction involved in illustrating units by means of them leaves no distinguishing properties. Hence in this context, Schröder means qualitative identity, which corresponds to sense 4 of '*gleich*'

[29] I will use 'oners' because Schröder prefers 'einer' to 'eins' and because it will help avoid confusion between the numeral one and the (or a) number one.

[30] In a collection, the oners are separated from each other by plus signs (p. 5), a representation of number he calls their '*Urform*' (p. 22). Note that Schröder says that the sum of oners is a natural number, not that it has a natural number. Schröder holds that the numbers consisting of distinct collections of oners are themselves distinct even when they are equipollent. In this respect, his *Zahl* resembles the first interpretation of Greek *arithmos* mentioned in footnote 11 above.

listed above, and we are left with a general representation of pure units.[31]

It is important to note that this maximal abstraction is not the result of the traditional understanding of abstraction that might, for example, proceed from the concept of a Clydesdale to the concepts of horse, animal, extended body, and being. One of the most interesting features of Frege's predecessors is that they depart from the traditional conception of abstraction to make room for a numerical kind of abstraction that suits the needs of mathematics. Unfortunately, Schröder does not further describe or defend the nature of this numerical abstraction. For the sake of argument let us grant that through some sort of numerical abstraction, we can get to a collection of oners.

Schröder's account lays claim to a completely general representation of units while simply assuming that those units remain distinguished. Since the maximal abstraction is intended to leave behind all distinguishing properties, however, there is nothing left to render the units distinct. Hence, Schröder does not provide a solution to the pure plurality problem. As Kant might put it, Schröder attempts to have his cake and eat it too.[32]

[31] One might think that in Schröder's view, we could take the oners themselves as pure units, that is, that we could represent them while abstracting from all their distinguishing properties other than their spatial relations, in the way that I have suggested Kant may think of using dots on a page. It is not entirely clear what the illustration relation consists in, but I think that on the most sympathetic reading of Schröder's position consistent with his claims of maximal abstraction, the oners represent pure units. As noted in §1 above, there are two sorts of purity at stake in the representation of pure units: a lack of distinguishing characteristics among a collection of units, and a lack of any distinguishing characteristics at all. Oners may have (or at least approximate) the first sort of purity, while clearly failing to achieve the second. See footnote 32 for more on oners and the first sort of purity. I would like to thank Charles Parsons, whose comments led me to distinguish the oners and the units they illustrate.

[32] It is possible that the distinctness of the oners is intended to represent the distinctness of units illustrated by means of them (see footnote 31). In practice, we distinguish between strokes or between tokens of the numeral '1' by their location in space and time, not by any qualitative differences, such as imperfections or the darkness of their ink. It is possible that Schröder may make at least implicit appeal to space to distinguish the numeral tokens, thereby implicitly adopting the Kantian position. Nevertheless, he nowhere states that a necessary condition of reaching a general representation of number is distinguishing between oners by spatial or temporal location alone. He does say that the fact that number

Stanley Jevons presents a quite different approach in *The Principles of Science* of 1874.[33] Jevons states that 'with difference arises plurality' and that '[N]umber is but another name for *diversity*' (175). Jevons distinguishes between concrete number and abstract number; the former corresponds to *arithmoi* and *Anzahl*—for example, three horses—while the latter refers to numbers *simpliciter*, like the number three.

Unlike Schröder, Jevons explicitly distinguishes numerical abstraction from logical abstraction. The latter gives us a conceptual representation of an abstract quality, as when we abstract from several objects that are red to the conception of the quality redness (33). In this sort of abstraction, we 'drop out of notice the very existence of difference and plurality' among the red objects (178). In contrast, numerical abstraction consists in abstracting from 'the character of the difference from which plurality arises, retaining merely the fact' (177; see also 178). In keeping with this conception of numerical abstraction, he describes abstract number as merely '*the empty form of difference* (177)'. Jevons holds that leaving out both the kinds of objects numbered and any specification of their differences while retaining the fact of their differences accounts for the generality of number. He is explicit that the notions of both concrete and abstract number specify that the units counted are to be distinguished, while leaving unspecified what characteristics will distinguish them (177–8). In short, Jevons explicitly embraces a concept of number that appeals to *indeterminate* distinguishing characteristics, and he singles out a special form of abstraction that results in such a concept. We seem to have found a solution to the difficulty reconciling the generality of number with the need for distinguishing characteristics, a solution that does not require the representation of pure units, and hence avoids the pure plurality problem.

Jevons' solution is not without its own difficulties, however. It is essential to the generality of concepts that they represent indeterminately, that is, that they need not specify all the properties an object might have. But this is a different sort of indeterminacy from that which Jevons proposes. For example, the concept of being a shape requires that an object have specific characteristics, being completely bounded in space, say, while it simply leaves out the characteristics

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consists of spatially separated components that can be generated at any time brings advantages (p. 9), but that is as far as he goes.

[33] Jevons (1874). All references to Jevons here below will be to this text (page numbers in brackets).

corresponding to a particular shape. In contrast, a concept that results from Jevons' numerical abstraction retains the fact of difference, which requires that each unit must have the characteristic of *having (or lacking) one or more characteristics that, in coordination with the one or more characteristics that each of the other units have (or lack), is sufficient to distinguish it from each of the other units*. The traditional understanding of conceptual representation, according to which characteristics simply represent particular qualities, simply does not make room for this sort of indeterminacy.

One might well respond: so much for the traditional understanding of a concept, which is also inadequate for other sorts of conceptual representation, and I would agree. What we are interested in, however, is how matters were viewed by Kant and others up to and including Frege. In general, Kant's discussions of concepts reflect a traditional understanding of the constitution of concepts. It seems likely that Kant at least implicitly rules out Jevons' approach, and hence Jevons' solution to the pure plurality problem.

The last text I will consider is Johannes Thomae's 1880 edition of *Elementare Theorie der Analytische Functionen*, which begins with the assumption that we can count.[34] He states that counting rests on our ability to abstract from the peculiarity [*Eigentumlickheit*] of the individuals of a collection of objects and then give successive names to different collections of such objects. He thinks that we can abstract from all peculiarity of objects without collapsing the collection. As a result, each unit is represented as equal to every other, and the concept of number arises from the counting of such units (1). In his view, the equality of units consists in their substitutability, which in turn consists in their lack of distinguishing properties after abstraction. Significantly, however, Thomae also states that abstraction does not remove the differences in spatial or temporal location of the units (1). Thus, Thomae holds the view I have attributed to Kant and is found in Schultz, according to which spatial and temporal location allow us to represent otherwise indistinguishable units and thereby contribute to solving the pure plurality problem.[35]

[34] Thomae (1880/1898). All references to Thomae will be to this work (with page numbers in brackets).

[35] Thomae significantly revised his account of number concepts in the 1898 edition, greatly expanding it and attempting to take into account the work of Dedekind and Frege, and in particular Frege's criticisms. He seems not to have really grasped the importance of Frege's advances; he still holds to a units view based on the purported empirical abstraction found in childhood concept acquisition, and he even states that his new

3. Frege and Kant on the limits of conceptual representation

I would now like to turn to Frege's arguments in the *Foundations of Arithmetic* against the possibility of representing units in a way that allows concepts of number to be based on them.[36] There are two terminological obstacles to understanding him. First, Frege rejects any conception of a number *n* as simply a collection of *n* things or as a property of a particular collection of *n* things.[37] It is therefore not surprising that Frege uses '*Zahl*' only for individual numbers. As we saw above, '*Anzahl*' was commonly used for collections of things rather than individual numbers, but Frege departs from common usage and often uses '*Anzahl*' as a synonym for '*Zahl*', which widens the distance between him and those he attacks.[38]

Second, Frege's use of '*gleich*' and its cognates raises particular difficulties. As noted in the previous section, '*gleich*' is multiply ambiguous. Because Frege treats numbers as singular objects and rejects a notion of a number *n* as a collection of *n* things or a property of a particular collection of *n* things, by his lights, it strictly speaking makes no sense to say that the number of planets is *equal* to the number of coins in my pocket; if it is true, then the number of planets is *one and the same* number as the number of coins in my pocket. Both senses, however, can be expressed by '*gleich*'.

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account does not differ substantially from Frege's. Nevertheless, he makes interesting further points relevant to the units view, which I will unfortunately not be able to explore here.

[36] Frege has various arguments undermining an appeal to units, including arguments that 0 and 1 cannot be accounted for and that arithmetical operations cannot be made to correspond to the union and dissolution of units, but most of his attention is directed against the possibility of representing the units and it is on this argument that I will focus.

[37] Frege does think of a cardinal number *n* as the object that is the collection of concepts under which exactly *n* things fall, but that is another matter.

[38] See, for example, the headings preceding §55 and the heading to §62. I noted above that Frege does not acknowledge the distinct accounts Schröder gives of *Anzahl* and *Zahl*; this may reflect Frege's conviction that there is no distinct account to be given. I am indebted to William Tait (2005), 38–9 and 242–3, for a helpful treatment of Frege's view of number, identity, and equality in contrast to the views of those Frege discusses. Tait believes that Frege's rejection of the conception of number as a collection leads him to misread Euclid, Hume, and Schröder; see footnote 40 below for a discussion of Hume and footnote 42 below for a discussion of Schröder.

Similarly, '*Gleichung*' is usually translated as 'equation,' which connotes an equality, but Frege maintains that an equation expresses a strict identity. In order to make Frege's views clear, Austin translates '*gleich*' as identical and '*Gleichung*' as identity.[39]

Unfortunately, this leads to misunderstandings. Austin translates '*gleich*' as 'identical' even where Frege is quoting or referring to the views of others, when they certainly do not mean one and the same, but mean something less than strict or numerical identity.[40] Any translation of these terms is beset with perils, of course. Nevertheless, translating '*gleich*' as 'identical' can lead to a misleading portrayal of the philosophers Frege discusses, if one has in mind strict or numerical identity.

There is a further issue about the meaning of '*gleich*' that is of even greater importance. As noted in the previous section, '*gleich*' is used for both equality between collections of things (i.e., equipollence) and for the purported requirement of equality or sameness among the distinct elements of a collection. It is crucial to distinguish between them. In the latter case as in the former, if we translate '*gleich*' as 'identical,' we cannot understand 'identical' to mean one and the same, or we will again misconstrue what Frege's contemporaries had in mind. In fact, in the latter case, we will make complete nonsense of their claims, because it is an obvious contradiction for elements of a collection to be both distinct and one and the same.

Now, 'identical' is itself ambiguous (as is reflected in the list of senses of '*gleich*' in the previous section). It can mean one and the same, that is, strict or numerical identity, which is probably the primary way a philosopher hears it today. But in a looser sense

[39] See Austin's (1950) English translation of Frege (1884), p. II, note.

[40] For example, when Hume articulates what has become known as 'Hume's Principle,' he states that those numbers (i.e. collections of things) that can be put into a one-to-one correspondence are called equal. When Frege discusses Hume's Principle in §63, he uses the term '*gleich*,' and goes on to discuss '*Gleichheit*'. As Tait (2005), p. 239, has argued, Frege seems to misread Hume here. Hume is using 'number' to refer to collections, and the relevant notion of *gleich* is of equality between collections, but Frege seems to take Hume to use 'number' to refer to singular objects and to interpret '*gleich*' as identity. In that case, translating '*gleich*' as 'identity' would capture Frege's misunderstanding of Hume, but it would not be true to what Hume said or meant. In this case, Austin translates '*gleich*' as 'equal,' since that was Hume's original term, and then translates Frege's use of '*Gleichheit*' as 'equality or identity'—the former term is a nod to what Hume in fact said, while the latter is how Austin thinks Frege interpreted him.

it can also mean that two or more things are qualitatively identical, that is, they share all of the same inner properties while remaining distinct—for example, the possibility of two identical rain drops, corresponding to sense 4 of '*gleich*' listed above. In this context, identical units are pure units. Thus, Austin's translation of '*gleich*' as 'identical' allows Frege's contemporaries to mean qualitatively identical when discussing the identity among distinct units of a collection.[41]

What is also important, however, is to allow that Frege himself uses 'identical' in the looser sense when he describes the views of his contemporaries, and not in the sense of strict identity, which he intends for equations or the number belonging to the concepts of equinumerous collections. For if we read him as meaning strictly identical, then he will both misrepresent them and make their views manifestly absurd. It is true that Frege relishes making his opponents' views appear absurd, but to do so by misreading their use of '*gleich*' as strictly identical would be willfully tendentious and weaken his attack. Indeed, if he understood '*gleich*' as strictly and numerically identical, the question that composes the heading to §34—'Are units identical [*gleich*] with one another?'—would have an immediate answer: it is logically impossible for a plurality of units to be one and the same unit. It is far better to understand Frege as allowing the looser, qualitative sense of identity in his attack and as arguing that his opponents' views lead to absurdity.[42]

[41] It may be that Austin only has the strict sense of identity in mind, for he seems to have some compunction about rendering Frege's description of Thomae's use of '*gleich*' as 'identical.' In §34, Frege states: 'Thomae calls the individual member of his set a unit, and says that "units are identical to each other" . . .'. Austin usually stays very close to the original, but in this case he drops the quotation and renders it: 'Thomae . . . says *in so many words* that units are identical with each other . . .' (my italics). If Austin also had the looser sense of identity in mind, he could have translated Frege more directly. This, of course, assumes that Thomae had this looser sense of identity in mind, as I argued above, but perhaps Austin was unsure of Thomae's meaning.

It is worth noting that in §34, Frege quotes three passages from Thomae and Lipschitz in a way that suggests the first and second passage are from Thomae and the third from Lipschitz. Austin breaks up the sentence and clearly attributes the second passage to Thomae, although it in fact belongs to Lipschitz.

[42] Tait (2005), p. 242, maintains that Frege understands '*gleich*' to mean strict identity, and that Frege imposes this reading on those he discusses. In Frege's §36 discussion of Hume's principle, where '*gleich*' is applied to equipollent collections, I think he is most likely correct; see footnote 40 above. Tait does not, however, explicitly distinguish between this use of '*gleich*'

Frege's argument strategy is to articulate what would be required for the unit view to succeed and to argue that those requirements cannot be met. He holds that if one is going to appeal to units, then the units must be pure, and that at least tacit recognition of this requirement motivates both the use of the maximally neutral word 'unit' and the appeal to identity (§34). Thus Frege and Kant are in complete agreement on the purity requirement. Moreover, both would agree that if someone, for example Lipschitz, does not acknowledge the need for pure units, then he is overlooking the requirements of a completely general representation of number by means of units.

Frege next argues that the units would also need to be distinct from each other, for where there is no diversity, there is no number (§35). Finally, he asserts that you cannot have the required sameness of units and a diverse collection of units at the same time. He cites Jevons, who refers to tokens of the numeral '1' with hash marks to insure that each refers to a distinct unit, and then states in his characteristically crisp polemic:

> The symbols
>
> $$1', 1'', 1'''$$
>
> tell the tale of our embarrassment. We must have identity [*Gleichheit*], hence the 1; but we must have difference—hence the strokes; only unfortunately, the latter undo the work of the former (§36; see also §38).

Frege asserts that if we were to derive the concept of number from units, they would have to be identical and diverse at the same time, which is impossible. The reference to impossibility might suggest that Frege has strict identity in mind. But in that case, when Frege states that we 'must have identity', he is certainly not speaking for Jevons or any other adherent of the pure units view, who would only claim that we require qualitative identity. In fact, as mentioned above, this would make Jevons' view patent nonsense. If, on the other hand, Frege instead means qualitative identity, then he appears to hold that qualitative identity would lead to strict identity, which in turn excludes diversity.

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and its application to elements among a collection, and he does not explicitly consider the possibility that Frege uses 'identity' in the looser sense when describing the purported identity of units among a collection that his opponents, such as Schröder, espouse.

This is similar to the position that Kant adopts when he considers objects as conceived by the understanding alone: you cannot represent the required units in their purity and represent them as diverse if you represent them merely by means of concepts. Thus, if we focus on Kant's views on representing by means of concepts alone, Frege and Kant agree that the requirements for representing pure units leave one in a bind (I will consider what each would think about appealing to space or time below). Both would claim that Schröder implicitly appeals to a notion of pure units without paying the price of abstracting from all distinguishing characteristics; hence, they would agree that Schröder is trying to have his cake and eat it too, and has not solved the pure plurality problem.

Let us look more closely at why Frege thinks that the units view is untenable. He describes the core of the difficulty with indistinguishable units in §36, where he says that units deprived of their distinguishing characteristics would collapse into one and be numerically identical (§39). Frege appears to be assuming some version of Leibniz's principle of the identity of indiscernibles. Section 34 provides evidence that he does; Frege asks why, in the view he is attacking, we ascribe identity to objects that are to be numbered. Frege seems to allow that identity comes in degrees and can be incomplete, which suggests that he is referring to qualitative identity. He then rules out the complete identity of two objects. Hence, Frege appears to assert that complete qualitative identity entails strict identity—the principle of the identity of indiscernibles.

Further evidence is found in §65, in which Frege states that all the laws of identity are contained in universal substitutability. The context makes it clear that Frege is here talking about strict, numerical identity. But if substitutability is the mark of strict identity, then it seems that qualitative identity would entail numerical identity, and hence some version of the principle of the identity of indiscernibles.[43]

This strengthens the parallel to Kant's views. As noted above, Kant states that if we only represented objects by means of concepts, then the principle of the identity of indiscernibles could not be disputed. And Kant, like Frege, describes the consequences as a kind of collapsing of the distinct units into one (§39).

There is one more important point of agreement between them. At several places in the *Foundations*, Frege describes traditional abstraction resulting in a general concept. In §34, for example, he states that if we disregard the distinguishing properties of a white cat and

[43] This would be so if qualitative identity implied substitutability. It is on the basis of §65 that Tait (2005), p. 235, ascribes the principle of identity of indiscernibles to Frege.

a black cat, our abstraction arrives at the concept 'cat' (see also §§44, 45, and 48). There are disagreements about Frege's views on abstraction, but we need not resolve whether Frege endorses traditional conceptual abstraction or merely concedes it for the sake of argument. The important point is that, even if he introduces traditional abstraction only for the sake of argument, he does so to contrast it with and to rule out a special form of abstraction that would result in number (e.g., §44). Moreover, Frege explicitly rejects the conception of numerical abstraction put forth by Jevons, that is, the possibility of abstracting to a concept which, in Jevons' terms, 'retains the fact of difference,' and 'implies the existence of the requisite differences' without specifying them.

Why does Frege reject Jevons' suggestion? He quotes Jevons' claim that numerical abstraction 'consists in abstracting the character of the difference from which plurality arises, retaining merely the fact'. Frege then states that if we attempt to abstract from the distinguishing characteristics '(. . .) we should never get so far as to distinguish the things at all, and consequently could not retain the fact of the existence of the differences either (. . .)' (§44).[44] Frege is reasserting his view that abstracting from distinguishing properties would lead to a collapse of the units into one. In other words, he rules out the possibility of characteristics that simply 'retain the fact of difference'.

Frege might be read as objecting to the process of numerical abstraction itself. Even if it is only for the sake of argument, however, Frege allows traditional conceptual abstraction in order to distinguish it from numerical abstraction, so what rules out numerical abstraction in particular? It seems to me unlikely that Frege has a worked out theory of mental activity that would distinguish between the former and the latter. More importantly and decisively, his earlier objection in §39 against units that are pure and yet distinct does not mention abstraction at all; nor do his discussions of Jevons in §36 and §38, where he focuses on the impossibility of units being pure yet distinguishable. He seems to think that the problem is not with the means of attaining a representation of pure units, but with the purported pure units themselves; in arguing against Jevons, he simply does not allow for Jevons' kind of characteristics. Thus, Frege seems to agree with Kant in rejecting characteristics that specify further characteristics indeterminately. It would seem that both Frege and Kant reject Jevons' solution to the problem for the same

[44] Frege's argument is somewhat more complicated than this summary suggests; he presents a dilemma, and I am drawing out what he says concerning the first horn.

reason: a shared assumption concerning the limits of conceptual representation.[45]

4. Frege and Kant on the role of space and time

I would now like to turn to Frege's main argument that we cannot overcome the limits of conceptual representation by appealing to space or time to represent pure units. His argument presses a dilemma (§41). The first horn concerns the possibility of representing points of space as qualitatively identical. Frege states that only in themselves, apart from their relations, are points of space qualitatively identical. But, he adds, 'if I am to think of them together, I am bound then to consider them in their collocation in space, or else they fuse irretrievably together into one'. Thus, points of space in themselves may be qualitatively identical, but they cannot be represented as distinct.

It seems that the homogeneity of points of space might support the claim that, in themselves, they are qualitatively identical. What, however, is the basis for Frege's claim that these qualitatively identical points will fuse into one? It appears that Frege either implicitly appeals to the principle of identity of indiscernibles, or thinks that this is a self-evident property of points of space, one that could be described as a case of the identity of indiscernibles.[46] Regardless,

[45] It is perhaps surprising to attribute to Frege a limitation on conceptual representation; his views on what is conceptually representable would appear to be, if anything, quite liberal. Nevertheless, it seems to me that Frege's argument points in this direction.

[46] Asserting a version of PII applied to points of space would be a departure from the traditional understanding of the principle. Leibniz only applies PII to substances or real beings and their phenomenal manifestations: monads, 'pieces of matter,' atoms, and sensible things such as leaves and drops of water. In his view, PII does not apply to abstractions or ideal beings, and because he holds that spaces and times are merely ideal, he does not think that PII applies to them (as Tait (2005), fn. 29, points out). In fact, Leibniz states that points of space and time are distinct despite being indistinguishable, and he even compares them to pure units in §27 of Leibniz's Fifth Paper of the *Leibniz–Clarke Correspondence*; see H. G. Alexander (1956). Kant would also reject the application of PII to points of space, but for a different reason. In his view, PII does not apply to *any* objects of sensibility, whether or not they are substances, and that includes qualitatively identical parts of space and points of space. Kant focuses on parts of space. In his view points are limits of spaces and hence

Frege simply seems to assume that there is no possibility of a primitive distinctness among points of space in themselves. One might think, for example, that points of space, though qualitatively identical, have *haecceities* (or something analogous) that distinguish them one from another. This would be a contentious claim, of course. Nevertheless, the success of his main line of argument against the units view depends on his ability to close off the possibility of appealing to space and time, and without further argument for his assumption, his main line of attack is threatened.[47]

Despite these worries, however, the heart of Frege's argument appears on the second horn of the dilemma, where he claims that points of space can only be distinguished by their relations.[48] He adds:

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presuppose parts of space, and the same diversity without specific difference holds for them as for qualitatively identical parts of space.

[47] Frege's statement concerning points of space reads:

> It is only in themselves, and neglecting their spatial relations, that points of space are identical to one another [*einander gleich*]; if I am to think of them together, I am bound then to consider them in their collocation in space, or else they fuse irretrievably into one (§41).

The phrases '*neglecting* their spatial relations' and 'if I am to *think* of them together [zusammenfassen]' and 'I am bound to *consider* them' leave open the possibility that Frege intends to be discussing *only our representation* of points of space rather than points of space themselves. This would be consistent with his argument strategy; that is, with his attempt to show that Schröder, Jevons, and Thomae cannot *represent* units as both pure and distinct, and hence cannot account for our *representation* of number by appealing to our representation of units. If Frege is making a point about our representation of points of space, then he may be implicitly invoking an intensionalized version of PII—that is, if someone represents x and y as indistinguishable, then that person represents x and y as identical. Nevertheless, as noted in the previous section, Frege asserts an un-intensionalized version of PII in §34, and §65 seems to support it as well. On balance, I think that it is likely that Frege implicitly applies un-intensionalised PII to points of space in §41. I would like to thank Walter Edelberg for suggesting intensionalized versions of PII to me and pressing me for clarification. Un-intensionalized and intensionalized versions of PII are closely related to each other in both Leibniz and Kant; there is much more to say on this point than I can include in this paper.

[48] One can question whether Frege's position makes sense. William Tait has argued that relations between points of space will not distinguish those points unless the relations themselves can be distinguished, which

> All these [spatial and temporal relations] are relationships which have absolutely nothing to do with number as such. Pervading them all is an admixture of some special element, which number in its general form leaves far behind (§41).

In short, Frege argues that the spatial relations that distinguish pure units will undermine the generality of the representation of number, because this spatiality will have to be represented in the concept of number.[49]

How would Kant respond? First, I think that Kant agrees with Frege that qualitatively identical parts and points of space are distinguished by their relations to one another.[50] Thus, despite qualms about Frege's handling of the first horn of the dilemma, I believe Kant would find himself on the second horn. Second, Kant would deny that the relations distinguishing qualitatively identical parts of space must be represented in the concept. Kant makes room for the diversity of pure units despite their indistinguishability by, in effect, drawing a distinction between conceptual and intuitive distinguishability. The former is all that is required for the purity of pure units: conceptual indistinguishability gives us the required generality, while intuitive distinguishability accounts for the diversity of these units.

Frege might reject Kant's attempt to avoid importing a 'special element' into the concept of number by appealing to intuition. He holds that *any* means sufficient to distinguish units, whether it be spatial relations or something else, would have to be specified in the concept of their collection, and hence in the concept of number. The nature of the distinguishing property is immaterial; either it is represented in the concept or it is not. On the one hand, Kant appeals to spatial relations among parts of space to distinguish units, and on the other, he simply omits this difference from the

presupposes the distinctness of points of space. Yet Frege has just denied that points of space in themselves are distinct, see Tait (2005), p. 235. If an adequate response to this criticism can be given, it will depend upon difficult issues concerning the nature of relations, distinctness, and identity, issues that I cannot adequately address in this paper.

[49] In fact, he holds that even a single point of space has something *sui generis* that distinguishes it from, say, a moment in time, and 'of which there is no trace in the concept of number' (§41).

[50] Much more can and needs to be said on this point, but I will have to postpone an explication and defense for another occasion.

concept. In short, Frege might say that Kant himself is trying to have his cake and eat it too.

This criticism simply underscores the differences between Frege and Kant's understanding of conceptual representation and its limits. On the criticism proffered on Frege's behalf, anything representable is in principle conceptually representable, while for Kant this is not the case, and intuitive representation can supplement conceptual representation. There is a further point brought out by the criticism, however. For in Kant's view, the generality of the representation of number is not undermined by supplementing conceptual representation with intuition. Whether Kant is entitled to this claim depends on a deeper analysis of the role of schemata in accounting for the generality of mathematical claims, as well as the role of pure units in allowing particular number concepts to arise and in establishing their objective validity in a way that does not undermine that generality. An answer to the questions raised on Frege's behalf will constrain an acceptable Kantian account of arithmetical cognition.

5. Summary

The idea that arithmetical concepts require the representation of units, and pure units in particular, arose as long ago as Plato, but its history effectively ends with Frege. Barring an overthrow of more than a century of mathematical logic, that is as it should be. Nevertheless, interest in the history of the foundations of mathematics requires that we understand this approach, the pure plurality problem, and attempts to overcome this problem. There is evidence that Kant assigns a role to the representation of pure units in the origin of particular number concepts and in establishing the objective validity of arithmetical claims. His account, however, is not subject to the pure plurality problem because he appeals to pure intuition. Understanding the views of Frege's contemporaries requires untangling ambiguous notions of equality and identity, but doing so reveals attempts to make the units view work by developing the notion of numerical abstraction and loosening the traditional understanding of conceptual representation. Clarifying their views and Frege's own notions of equality and identity allow us to better understand his arguments against them.

A comparison of Kant and Frege on pure units also uncovers unexpected features of their views. They are in closer agreement on the units view than one might have expected, and would criticize it on grounds of the limits of conceptual representation. Yet they deeply diverge on whether an appeal to space and time could make the representation of pure units possible. The core issue dividing them is whether space and time make it possible to distinguish units in a way that does not undermine the generality of the concept of number. Frege claims that they will not, but he assumes that space and time lack primitive distinctness, and his claim requires further argument. Kant thinks that they will, but whether he is entitled to this claim depends on his fuller account of the generality of mathematical cognition.[51]

[51] Parsons (1984) explored Kant's conceptions of magnitude and their relation to mathematical cognition and to the categories of quantity. Friedman (1992a), 110–113, drew a connection between Kant's account of algebra and the Eudoxian theory of proportions and he subsequently encouraged me to explore the connection between the Eudoxian theory and Kant's theory of magnitudes. My work can thus be viewed as an extension of Parsons' and Friedman's lines of investigation and I have benefited greatly from their work. Discussions with William Tait revealed that he had addressed some of the same issues I was exploring. I have benefited immensely from Tait's (2005) paper and from regular discussions with him in the spring of 2007 and 2008. I would also like to thank those who responded to versions of this paper delivered at the Central Meeting of the American Philosophical Association, April 2007; the *Kant and Philosophy of Science Today* conference at the Royal Institute of Philosophy, University College London, July 2007; and the UCSB Department of Philosophy, February 2008. I am particularly grateful to Michael Friedman, Robert Howell, and Charles Parsons for detailed comments, as well as Brandon Look for comments and for sharing a draft of his paper on Leibniz's principle of identity of indiscernibles. Special thanks are due the members of the UIC Philosophy of Mathematics Reading Group, especially Bill Hart and Walter Edelberg, for their many insightful remarks, which substantially improved both my understanding of the philosophical issues and this paper. I am also grateful to Marcus Giaquinto for discussion and for comments on the penultimate draft. Finally, I would like to acknowledge the generous support of the National Science Foundation; this paper is based upon work supported by them under Grant No. 0452527.