Reading the Begriffsschrift[1]

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The aim of the third part of the *Begriffsschrift*, Frege tells us, is:

> to give a general idea of the way in which our ideography is handled . . . Through the present example, moreover, we see how pure thought, irrespective of any content given by the senses or even by an intuition a priori, can, solely from the content that results from its own constitution, bring forth judgements that at first sight appear to be possible only on the basis of some intuition . . . The propositions about sequences developed in what follows far surpass in generality all those that can be derived from any intuition of sequences. If, therefore, one were to consider it more appropriate to use an intuitive idea of sequence as a basis, he should not forget that the propositions thus obtained, which might perhaps have the same wording as those given here, would still state far less than these, since they would hold only in the domain of precisely that intuition upon which they were based.[2]

He then proceeds to give a definition, proposition 69, on which he comments, 'Hence this proposition is not a judgement, and consequently *not a synthetic judgement* either, to use the Kantian expression. I point this out because Kant considers all judgements of mathematics to be synthetic.'[3]

In the preface to the *Begriffsschrift* he states, 'To prevent anything intuitive [*Anschauliches*] from penetrating here unnoticed, I had to bend every effort to keep the chain of inferences free of gaps.'[4]

It is evident from the anti-Kantian tone of these remarks that Frege regards himself as showing the inadequacy of a certain (unspecified) Kantian view of mathematics, by supplying examples of judgements that he thinks 'at first sight appear to be possible only on the basis of some intuition', but which pure thought, 'solely from the content that results from its own constitution', can bring forth. However an exact statement of the Kantian position under attack might run, the view is one according

[1] I am grateful to Michael Dummett, Robin Gandy, Daniel Isaacson, David Lewis, and the editor for helpful comments. This paper was written while I was on a Fellowship for Independent Study and Research from the National Endowment for the Humanities.

[2] Gottlob Frege, *Begriffsschrift, a Formula Language, Modeled Upon That of Arithmetic, For Pure Thought*, p. 55. All references are to the Bauer-Mengelberg translation, found in *From Frege to Gödel: A Source Book In Mathematical Logic*, ed. Jean van Heijenoort, Harvard University Press, Cambridge, Massachusetts, 1967.

[3] Ibid. The remark that Kant considers all judgements of mathematics to be synthetic seems somewhat intemperate: Kant might of course agree that 69 is no *judgement*, hence no synthetic judgement.

[4] Op. cit., p. 5.

to which no non-trivial mathematical judgement is 'possible' without 'a priori intuition'.

My principal aim in this paper is to examine Frege's procedure in the third part of the *Begriffsschrift* in order to see how, and how well, a Kantian view of Frege's examples might be defended and to determine to what extent Frege could claim to have shown the truth of a view that may be called *sublogicism*: the claim that there are (many) interesting examples of mathematical truths that can be reduced (in the appropriate sense) to logic. Inevitably, the uncertainties and obscurities attaching to the notions of *intuition* and *logic* will leave these matters somewhat unresolved. I will, however, argue that a compelling case for Frege's view can be made against a certain sort of defence of Kant.

The issue between Frege and Kant is joined over a certain technical point that arises in connection with the marginal annotations of the derivations of part 3. If we wish to understand the issue, we cannot avoid examining the wallpaper. There is a further reason for looking at the formalism of part 3: at least one little-known but major master-stroke is hidden there, and one of the subsidiary aims of this paper is to call attention to it, repellent though the notation in which it is cloaked may be. Another aim of the paper is simply to render part 3 more accessible.

Before we examine Frege's achievement, we must review the special notational devices which Frege introduces in part 3. Fortunately, there are only four of them.

The first of these '$\overset{\delta}{\underset{\alpha}{|}}\left(\begin{array}{l} F(\alpha) \\ f(\delta,\alpha) \end{array}\right.$', is defined in proposition 69 to mean something that we might notate: $\forall d \forall a(Fd$ & $dfa \rightarrow Fa)$. (I have written 'dfa' in place of Frege's '$f(d, a)$'.) Since the relation f—Frege calls it a *procedure*—is fixed throughout part 3, I shall use the abbreviation 'Her(F)', suppressing 'f', for this notion instead. ('Her' is for 'hereditary'.)

The second, '$\overset{\gamma}{\underset{\beta}{\sim}} f(x_\gamma, y_\beta)$', is Frege's abbreviation for the strong ancestral of f, whose celebrated definition is presented in proposition 76. Abbreviating '$\forall a(xfa \rightarrow Fa)$' as 'In($x, F$)' (again suppressing mention of the fixed f), we may give the definition as: $\forall F(\mathrm{Her}(F)$ & $\mathrm{In}(x, F) \rightarrow Fy)$. We shall use: $xf{*}y$ for this notion.

The third, '$\overset{\gamma}{\underset{\beta}{\overline{\sim}}} f(x_\gamma, z_\beta)$', is the abbreviation for the weak ancestral, defined in proposition 99 as: $xf{*}z \vee z = x$. We write this: $xf{*}_{=}z$.

Finally, Frege defines '$\overset{\delta}{\underset{\varepsilon}{\mathrm{I}}} f(\delta, \varepsilon)$' in proposition 115 to mean: $\forall d \forall e \forall a(dfe$ & $dfa \rightarrow a = e)$. We write this: FN (for 'f is a function').

We can now say what the judgements are which Frege thinks can be brought forth by pure thought solely from the content that results from its own constitution—or, as we may say, can be proved by purely logical means—but which, he thinks, appear at first sight to be possible only on the basis of some intuition. We can then take up the question whether the means used to prove them are in fact 'purely logical'.

If we look at the table with which the *Begriffsschrift* ends and which indicates which propositions are immediately involved in the derivations of which others, we find that there are only two propositions in the third part not used in the derivation of any others: number 98 and the last one, number 133. Since these propositions are not used to prove any others, I do not find it too far-fetched to suppose that Frege thought of these as illustrating the falsity of the Kantian view with which he is concerned.

The translation into our notation of 98 is: $xf{*}y \;\&\; yf{*}z \rightarrow xf{*}z$. That of 133 is: $FN \;\&\; xf{*}m \;\&\; xf{*}y \rightarrow yf{*}m \vee mf{*}_{=}y$.[5] These state that the (strong) ancestral is transitive and that if the underlying relation f is a function, then the ancestral connects any two elements m and y to which some one element x bears the ancestral. The analogy with the transitivity and connectedness of the less-than relation on the natural numbers, which is the ancestral of the relation *immediately precedes*, will not have escaped the reader's notice, and I dare say it did not escape Frege's.

Although Frege does not explicitly single out 98 and 133 as noteworthy in any way, it is quite reasonable to suppose that he regarded both of them as the sort of proposition that would justify the anti-Kantian viewpoint sketched above. For not only are these two the only propositions in part 3 not used in the demonstration of others, their content can be seen as a generalization of that of familiar and fundamental mathematical principles, for the grasp of whose truth some sort of 'intuition' was often supposed in Frege's time to be required. Moreover, one who attempts to convince himself of the truth of, for example 98, might well hit upon an argument that would seem to make appeal to the sort of intuition which Frege was concerned to show unnecessary. Suppose that y follows x in the f-sequence and z follows y. Then if one starts at x and proceeds along the f-sequence, one can eventually reach y. Ditto for y and z. Thus, by starting at x and proceeding along the f-sequence, one can eventually reach z, first by going to y, and thence to z. Thus z follows x in the f-sequence. Intuition, it might be suggested, discloses to us that any two paths from x to y and y to z can be combined into one single path from x to z: intuit them both and then attach in thought the beginning of the second to the end of the first. Or some such thing.

[5] I do not know why Frege chose to use the variable 'm' here instead of (say) 'w'.

The procedure Frege employs in the derivation of 98 is of considerable interest, and we shall look at its final steps. Having arrived at

84 $\mathrm{Her}(F)$ & Fx & $xf{*}y \rightarrow Fy$

and

96 $xf{*}y$ & $yfz \rightarrow xf{*}z$,

Frege generalizes upon z and y in 96 to obtain $\forall d\forall a(xf{*}d$ & $dfa \rightarrow xf{*}a)$. He then substitutes $\{a : xf{*}a\}$ for F (as we might put it) in the definition of $\mathrm{Her}(F)$ to obtain 97, which we can write: $\mathrm{Her}(\{a : xf{*}a\})$. He then reletters x and y as y and z in 84, substitutes $\{a : xf{*}a\}$ for F in 84, and discharges $\mathrm{Her}(\{a : xf{*}a\})$ to obtain the desired 98.

Frege appears to regard the substitution of a formula for a relation letter in an already demonstrated formula as on a par with substitution of a formula for a propositional variable or relettering of a variable. Of course, in standard first-order logic, substitution of formulae for relation letters gives rise to no special worries: any formula demonstrable with the help of substitution is demonstrable without it. (Frege performs several such substitutions in part 2, which contains none but first-order notions.) But this is emphatically not the case as regards part 3 of the *Begriffsschrift*. The capacity to substitute formulae for relation letters gives the whole of Frege's system, which is not a system of first-order logic, significantly more power than it would otherwise have.

Although a Kantian opponent could well make an objection at this point to Frege's use of substitution, there is a more pertinent objection to be made: no one can sensibly think that *every* mathematical judgement must be based on some intuition. For certainly there are some trivial mathematical judgements which need not be so based, among them analytic judgements concerned with mathematical matters and others of a *trivial* logical nature, such as '$5+7 = 5+7$' or 'if $5+7 = 12$, then $5+7 = 12$'. Moreover, among such judgements are those that follow from definitions with only a *small* amount of *elementary* logical manipulation. And one of these is Frege's 98. For, let us face it, Frege's proof of 98 is unnecessarily non-elementary. One needs no rule of substitution at all to prove that if $xf{*}y$ and $yf{*}z$, then $xf{*}z$. For suppose $xf{*}y$ and $yf{*}z$. We want to show $xf{*}z$, i.e. $\forall F(\mathrm{Her}(F)$ & $\mathrm{In}(x, F) \rightarrow Fz)$. So suppose $\mathrm{Her}(F)$ and $\mathrm{In}(x, F)$. We want to show Fz. Since $yf{*}z$, we need only show $\mathrm{In}(y, F)$, i.e. $\forall y(yfa \rightarrow Fa)$. So suppose yfa. Since $xf{*}y$, $\mathrm{Her}(F)$ and $\mathrm{In}(x, F)$, Fy. And since $\mathrm{Her}(F)$ and yfa, Fa, QED. The trouble with 98, our Kantian might complain, is that although the *above* proof of 98 is certainly a proof by logical means alone, 98 does not *look at first sight as if* it must be based on an intuition.

Frege has not yet laid a glove on the Kantian. 98 *is* a weak example. Of course Frege's rendering of 98, 'If y follows x in the f-sequence and

z follows y in the f-sequence, then z follows x in the f-sequence', might have been a better choice, but the Kantian might then have been in a position to raise questions about the grounds for *reading* '$xf*y$' as 'y follows x in the f-sequence', plausibly arguing that this reading is itself justified only on an intuition.

No such objection can be raised against 133, FN & $xf*y$ & $xf*m \rightarrow yf*m \vee mf*_{=}y$, of which an 'intuitive' proof might go as follows. Suppose FN, $xf*y$, and $xf*m$. Since $xf*y$ and $xf*m$, there is an f-sequence leading from x to y and an f-sequence leading from x to m. And since FN, each thing bears f to at most one thing; thus at no point along the way can either of these paths diverge from the other. Thus the paths coincide up to the point at which any shorter one gives out. Since $xf*m$ and $xf*y$, we eventually reach both m and y; when we have done so, we will evidently have reached y before m, reached m before y, or reached m and y at the same time. In the first case, we can get from y to m along the path obtained by removing the path from x to y from the path from x to m; in the second case, we can similarly get from m to y, and in the third case, $m = y$. Thus $yf*m \vee mf*_{=}y$. We are about to turn to Frege's derivation of 133; before we do so, the reader might like to try his hand at giving a proper proof of 133, in the style of the proof of 98 given two paragraphs above. (One such proof is given in the appendix.)

One significant landmark in Frege's derivation of 133 is proposition 110: $\forall a(yfa \rightarrow x*f = a)$ & $yf*m \rightarrow xf*_{=}m$. 110 is itself got from 108: $zf*_{=}y$ & $yfv \rightarrow zf*_{=}v$, which has a straightforward proof.[6] 108 is fairly obvious; 110 is not at all obvious. (We cannot get yfm from $yf*m$.) How does Frege get 110 from 108?

First of all, he reletters the variables in 108, replacing z, y, and v by x, (German) d, and (German) a, and then universally quantifies upon a and d to get: $\forall d \forall a(xf*_{=}d$ & $dfa \rightarrow xf*_{=}a)$. He then takes 75: $\forall d \forall a(Fd$ & $dfa \rightarrow Fa) \rightarrow \mathrm{Her}(F)$, which is one-half of the definition of $\mathrm{Her}(F)$, substitutes $\{a : xf*_{=}a\}$ for F (as we would put it), and uses 108 to cut the antecedent of the result, thereby getting 109: $\mathrm{Her}(\{a : xf*_{=}a\})$. Next he takes 78: $\mathrm{Her}(F)$ & $\forall a(xfa \rightarrow Fa)$ & $xf*y \rightarrow Fy$, which is a trivial consequence of the definition of the ancestral, respectively replaces x and y by y and m, again substitutes $\{a : xf*_{=}a\}$ for F, and drops $\mathrm{Her}(\{a : xf*_{=}a\})$ from the result by 109, to get $\forall a(yfa \rightarrow xf*_{=}a)$ & $yf*m \rightarrow xf*_{=}m$, as desired.

The complexity of the definition of the substituend $\{a : xf*_{=}a\}$ is noteworthy. '$xf*_{=}a$' abbreviates a disjunction one of whose disjuncts is a second-order universal quantification of a first-order formula. Were Frege merely substituting $\{a : Ga\}$ (G a one-place relation letter) for F, i.e. relettering F as G, we should have no qualms about his procedure.

[6] *Proof*: Assume $zf*_{=}y$, yfv, $\mathrm{Her}(F)$ and $\mathrm{In}(x, F)$. If $zf*y$, then Fy, and by yfv and $\mathrm{Her}(F)$, Fv; but if $y = z$, then $\mathrm{In}(y, F)$ and again Fv, as yfv. Thus $zf*v$, whence $zf*_{=}v$.

But the substitution of so complicated a formula as $xf*_{=}a$ for a relation letter is a matter considerably more problematical.

Having obtained 110, Frege straightforwardly gets 129: FN & $(yf*m \lor mf*_{=}y)$ & $yfx \rightarrow (xf*m \lor mf*_{=}x)$.[7] 131: $FN \rightarrow \text{Her}(\{a : af*m \lor mf*_{=}a\})$ follows, again by a substitution, this time of $\{a : af*m \lor mf*_{=}a\}$ for F in the quasi-definitional 75.

Frege then performs the same substitution to conclude the derivation. From 131, he uses propositional logic in infer 132: $[\text{Her}(\{a : af*m \lor mf*_{=}a\})$ & $xf*m$ & $xf*y \rightarrow (yf*m \lor mf*_{=}y)] \rightarrow [FN$ & $xf*m$ & $xf*y \rightarrow (yf*m \lor mf*_{=}y)]$. To get 133, the consequent of 132, he must obtain the antecedent. This is how he does it. He has earlier established 81: Fx & $\text{Her}(F)$ & $xf*y \rightarrow Fy$ (an easy consequence of the definition of the ancestral). By propositional logic there follows 82: $(p \rightarrow Fx)$ & $\text{Her}(F)$ & p & $xf*y \rightarrow Fy$. (Frege uses 'a' instead of 'p'.) He then substitutes hx for p and $\{a : ha \lor ga\}$ for F in 82 ('h' and 'g' are one-place relation letters, like 'F') and drops a tautologous conjunct of the antecedent to obtain 83: $\text{Her}(\{a : ha \lor ga\})$ & hx & $xf*y \rightarrow hy \lor gy$. The final logical move of the *Begriffsschrift* is the substitution in 83 of $\{a : af*m\}$ and $\{a : mf*_{=}a\}$ for h and g, which yields the antecedent of 132.

Of course, Frege could have condensed these two substitutions for F into one, by substituting $\{a : af*m \lor mf*_{=}a\}$ for F in 81 and using propositional logic to obtain the antecedent of 132. But to prove 133, Frege has had to make two essential uses of substitution, the first being the earlier substitution of $\{a : xf*_{=}a\}$ for F, the second, that of $\{a : af*m \lor mf*_{=}a\}$. It is noteworthy that the—or at any rate, one—obvious attempt to prove 133 will require the same two substitutions, in the order in which they are found in Frege's derivation.

The fact that the *Begriffsschrift* contains a subtle and ingenious double induction—for that is what Frege's pair of substitutions amounts to—used to prove a significant result in the general theory of relations is not, I think, well known, and the distinctively mathematical talent he displayed in discovering and proving the result is certainly not adequately appreciated. Frege's accomplishment may be likened to a feat the Wright brothers did not perform: inventing the airplane *and* ending its first flight with one loop-the-loop inside another.

Our Kantian has patiently had his hand up during this discussion of Frege's method in part 3, and it is time to give him his say.

The Kantian: 'I could not agree with you more about the excellences of proposition 133 and Frege's proof of it, but it is not a counterexample to

[7] *Proof*: Assume FN, $(yf*m \lor mf*_{=}y)$, and yfx. We must show $xf*m \lor mf*_{=}x$. Suppose $yf*m$. By 110 we need only show $\forall a(yfa \rightarrow xf*_{=}a)$, for then $xf*_{=}m$, whence $xf*m$ or $m = x$, and then $xf*m \lor mf*_{=}x$. So suppose yfa. Since yfx and FN, $x = a$, whence $xf*_{=}a$. Now suppose $mf*_{=}y$. We show $mf*x$, whence $mf*_{=}x$. Assume $\text{Her}(F)$ and $\text{In}(m, F)$. We are to show Fx. If $mf*y$, then since $\text{Her}(F)$ and $\text{In}(m, F)$, Fy, and then, since yfx and $\text{Her}(F)$, Fx. But if $y = m$, then from yfx, mfx, whence again Fx, since $\text{In}(m, F)$.

any thesis that I hold or that a reasonable Kantian ought to hold. Indeed, if anything, it is confirming evidence for my view. I agree that 133 is precisely the sort of proposition that is possible only on the basis of an intuition. But I disagree that Frege has been able to prove it without the aid of any intuition at all. In fact, the feature of Frege's method that you have been at pains to emphasize, the substitution of formulae for relation letters, is precisely the point at which, I wish to claim, Frege appeals to intuition. I'd be prepared to concede, for the sake of avoiding an argument, that nowhere in the rest of the *Begriffsschrift* is an appeal to intuition made. But I do wish to claim that his use of the rule of substitution does involve him in just such an appeal.

'The difficulty that the rule of substitution presents can best be seen if we consider the axiom schema of comprehension: $\exists X \forall x(Xx \leftrightarrow A(x))$. It is well known that in the presence of the other standard rules of logic, the substitution rule and the comprehension schema are deductively equivalent; given either, one can derive the other. In outline, the proof of this equivalence runs as follows. From the provable $\forall x(Fx \leftrightarrow Fx)$, we obtain $\exists X \forall x(Xx \leftrightarrow Fx)$ by second-order existential generalization, whence by the substitution of $\{a : A(a)\}$ for F, we have $\exists X \forall x(Xx \leftrightarrow A(x))$. Conversely, we observe that for any formulae $P[F]$ and $A(x)$, we can prove $\forall x(Fx \leftrightarrow A(x)) \rightarrow (P[F] \leftrightarrow P[\{a : A(a)\}])$; the demonstration of this is an induction on subformulae of the formula $P[F]$. Now suppose that $P[F]$ is provable. Then so is $\forall x(Fx \leftrightarrow A(x)) \rightarrow P[\{a : A(a)\}]$; and since the consequent $P[\{a : A(a)\}]$ does not contain F, $\exists X \forall x(Xx \leftrightarrow A(x)) \rightarrow P[\{a : A(a)\}]$ is also provable. Thus if we have as an axiom $\exists X \forall x(Xx \leftrightarrow A(x))$, as is guaranteed by comprehension, $P[\{a : A(a)\}]$ is provable too, QED. Thus we cannot admit substitution as a logical rule unless we are prepared to admit that all instances of the comprehension schema $\exists X \forall x(Xx \leftrightarrow A(x))$ are logical truths, and that is precisely what I wish to deny.

'For what does $\exists X \forall x(Xx \leftrightarrow A(x))$ say? If we look at the *Begriffsschrift*, we find that when Frege wishes to decipher his relation letters and second-order quantifiers, he uses the terms "property", "procedure", "sequence"; he uses the terms "result of an application of a procedure" and "object" to tell us what sorts of things free variables like "x" and "y" denote. My point can be put as follows. Suppose that $A(x)$ is the formula: $mf{*}x$. Then Frege would read the corresponding instance of the comprehension schema as "There is a property whose instances are exactly the objects that follow m in the f-sequence". This comprehension axiom is demonstrable in the *Begriffsschrift*. My question is: why should we believe that there is any such property? Now, *I* don't want to deny that there is such a property. I might well want to say that it's *obvious* or *evident* that there is one. And I would want to say to anyone who professed uncertainty concerning the existence of the property, "But don't you *see* that there has to be one?" In short, it is an intuition of precisely the kind Frege thinks he has shown unnecessary that licenses the rule of substitution. Thus Frege has not dispensed with intuition; he is up to his ears in it. (I may add that the inference from $\forall x(Fx \leftrightarrow Fx)$ to $\exists X\forall x(Xx \leftrightarrow Fx)$ also strikes me as problematical, but as it is legitimated by (the second-order analogue of) the standard logical rule of existential generalization, I have agreed not to object to it.)

'Moreover, there is an important difficulty connected with the interpretation of the *Begriffsschrift*.[8] Frege does not discuss the question whether properties are objects, as one might put it. It is uncertain whether Frege thinks there can, for example, be sequences of properties, whether xfy might hold when x and y are themselves properties. One would have supposed so; but then, of course, taking "f" to mean "is a property that is an instance of the property" produces a Russellian problem: $\exists X\forall x(Xx \leftrightarrow -xfx)$ is derivable in the *Begriffsschrift*, but would be read by Frege "There is a property whose instances are all and only those properties that are not instances of themselves", which is false, of course. Thus the system, although perhaps formally consistent, cannot be interpreted as Frege interprets it in the absence of some—I think the right word is "metaphysical"—doctrine of properties, which Frege does not supply. And what, pray, is the source of any such doctrine to be—pure logic? How then are we to interpret the *Begriffsschrift* so that its theorems all turn out to be truths that it does not require the aid of intuition to accept?

'I'm almost finished. Matters are no better and probably worse if Frege reads a second-order quantifier $\exists F$ as "There is a set F . . .". For sets clearly are "objects"; thus the difficulty presented by Russell's paradox immediately arises if we take the range of "F" to be all *sets*. The only escape that I can see for Frege is for him to stipulate that the *Begriffsschrift* is to be employed in formalizing a certain theory only if the theory does not speak about *all* objects. The rule of substitution would then be licensed by the *Aussonderungsschema* of set theory. But besides noting that this way out appears to be strongly at odds with his intentions in setting forth the *Begriffsschrift*, we may well wonder what justifies this appeal to the *Aussonderungsschema* if not *intuition* of some sort, for example the *picture* of the set-theoretic universe that yields the so-called "iterative conception of set". And now, I *am* finished.'

In reply: Russell's paradox does indeed show the difficulty of taking the second-order quantifiers of the *Begriffsschrift* as ranging over all sets or all properties and reading atomic formulae like Xx as meaning 'x is a member (or instance) of X'. We must find another way to interpret the formalism of the *Begriffsschrift*, on which we are not committed to the existence of such entities as sets or properties, and on which the

[8] For an illuminating discussion of this difficulty, see I. S. Russinoff, 'Frege's Problem about Concepts', MIT Ph.D. thesis, 1983.

comprehension schema $\exists X \forall x(Xx \leftrightarrow A(x))$ can plausibly be claimed to be a logical law.

Interpretation of a logical formalism standardly consists in a description of the objects over which the variables of the formalism are supposed to range and a specification that states to which of those objects the various relation letters of the formalism apply. Since Frege nowhere specifies what his relation letters 'f', 'F', etc. apply to, it is clear, I think, that he had no one 'intended' interpretation of the *Begriffsschrift* in mind: 'f', for example, will have to be interpreted on each particular occasion by mentioning the pairs of objects that it is then intended to apply to. But it appears that Frege did intend the first-order variables of the *Begriffsschrift* to range over absolutely all of the 'objects', or things, that there are. In any event, even if Frege did envisage applications in which the first-order variables were to range over some but not all objects, it seems perfectly clear that he did allow for some applications in which they do range over absolutely all objects. And because a use of the *Begriffsschrift* in which the variables do not range over all objects that there are can, by introducing new relation letters to relativize quantifiers, be treated as one in which they do range over all objects, we shall henceforth assume that the *Begriffsschrift*'s first-order variables do range over all objects, whatever an object might happen to be.

But what do the second-order variables range over, if not all sets or all properties? I think that a quite satisfactory response to this question is to reject it, to say that no separate specification of items over which the second-order variables range is needed once it has been specified what the first-order variables range over.[9] Instead we must show how to give an intelligible interpretation of all the formulae of the *Begriffsschrift* that does not mention special items over which the second-order variables are supposed to range and on which Frege's rule of substitution appears as a rule of logic and the comprehension axioms appear as logical truths.

The key to such an interpretation can be found in the behaviour of the logical particle 'the'.

If the rocks rained down, then there are some things that rained down; if each of *them* [pointing] is a K and each K is one of them, then there are some things such that each of them is a K and each K is one of them; if Stiva, Dolly, Grisha, and Tanya are unhappy with one another, then there are some people who are unhappy with one another. Existential generalization can take place on plural pronouns and definite descriptions as well as on singular, and existential generalization on plural definite descriptions is the analogue in natural language of Frege's rule of substitution. This type of inference is not adequately represented by the apparatus of standard first-order logic. However, a formalism like that of the *Begriffsschrift* can be used to schematize plural existential generalization, and our under-

[9] For more on this topic, see my 'Nominalist Platonism', *Philosophical Review*, 1985.

standing of the plural forms involved in this type of inference can be appealed to in support of the claim that Frege's rule is properly regarded as a rule of logic.

By a 'definite plural description' I mean either the plural form of a definite singular description, for example 'the present kings of France', 'the golden mountains', or a conjunction of two or more proper names, definite singular descriptions, and (shorter) definite plural descriptions, for example 'Russell and Whitehead', 'Russell and Whitehead and the present kings of France'.

Like the familiar condition: $\exists x \forall y(Ky \leftrightarrow y = x)$ which must be satisfied by a definite singular description 'The *K*' for its use to be legitimate, there is an analogous condition that must be satisfied by definite plural descriptions. In the simplest case, in which a definite plural description such as 'the present kings of France' is the plural form of a definite singular description, the condition amounts only to there being one object or more to which the corresponding count noun in the singular description applies. (Two or more, technically, if Moore and the Eleatic Stranger were right.) Thus like the definite singular description 'The *K*', which has a legitimate use iff the *K* exists, i.e. iff there is such a thing as the *K*, 'The *K*s' has a legitimate use iff the *K*s exist, i.e. iff there are such things as the *K*s, iff there is at least one *K*.

The obvious conjecture—I do not know whether or not it is correct—is that the general condition for the legitimate use of a conjunction of proper names, definite singular descriptions, and (shorter) definite plural descriptions is simply the conjunction of the conditions for the conjoined names and descriptions. We need not worry here whether the conjecture is true; for our purposes it will suffice to consider only definite plural descriptions of the simplest sort, plural forms of definite singular descriptions.

The connection between definite plural descriptions and the comprehension principle is that the condition under which the use of 'The *K*s' is legitimate, viz. that there are some such things as the *K*s, can also be expressed: there are some things such that each *K* is one of them and each one of them is a *K*. Thus 'if there is at least one *K*, then there are some things such that each *K* is one of them and each of them is a *K*' expresses a logical truth. Moreover, it is a logical truth that it is quite natural to symbolize as

$$\exists x Kx \rightarrow \exists X(\exists x Xx \ \&\ \forall x(Xx \leftrightarrow Kx)),$$

which is equivalent to the instance $\exists X \forall x(Xx \leftrightarrow Kx)$ of the comprehension scheme. Thus the idea suggests itself of using the construction 'there are some things such that . . . them . . .' to translate the second-order existential quantifier $\exists X$ so that comprehension axioms turn out to have readings of the form 'if there is something . . ., then there are some things such that each . . . thing is one of them and each of them is something . . .'. Let us see how this may be done.

We begin by supposing English to be augmented by the addition of pronouns 'it_x', 'it_y', it_z', . . .; 'that_x', 'that_y', 'that_z', . . .; '$\text{they}_X/\text{them}_X$', '$\text{they}_Y/\text{them}_Y$', '$\text{they}_Z/\text{them}_Z$', . . .; and '$\text{that}_X$', that_Y', 'that_Z', ... (For each first-order variable v of the formalism, we introduce 'it'$^\frown{}_v$ and 'that'$^\frown{}_v$; and for each second-order variable V, 'they'$^\frown{}_V$, which is sometimes written 'them'$^\frown{}_V$, and 'that'$^\frown{}_V$.) The purpose of the subscripts is simply to disambiguate cross-reference and has nothing to do with the distinction between first- and second-order formulae or between singular and plural number. A similar augmentation would be required for translation into English of first-order formulae of the language of set theory containing multiple nested alternating quantifiers, for example formulae of the form $\forall w \exists x \forall y \exists z R(w,x,y,z)$. The extension of English we are contemplating is a conceptually minor one, rather like lawyerese ('the former', 'the latter', 'the party of the seventeenth part'); our subscripts are taken for convenience to be the variables of the *Begriffsschrift* (instead of, say, numerals), but they no more *range over* any items than does 'seventeen' in 'the party of the seventeenth part'.

We now set out a scheme of translation from the language of the *Begriffsschrift* into English augmented with these subscripted pronouns.[10] Thus we specify the conditions under which sentences of the *Begriffsschrift* are true by showing how to translate them into a language we understand.

The translation of the atomic formula Xx is $\ulcorner\text{it}_x$ is one of $\text{them}_X\urcorner$. (The corner-quotes are Quinean quasi-quotes.)

The translation of the atomic formula $x = y$ is $\ulcorner\text{it}_x$ is identical with $\text{it}_y\urcorner$. The translation of any other atomic formula, for example Fx or xfy, is determined in an analogous fashion by the intended reading of the predicate letter it contains.

Let $F*$ and $G*$ be the translations of F and G. Then the translation of $-F$ is $\ulcorner$Not: $F*\urcorner$ and that of $(F \mathbin{\&} G)$ is $\ulcorner$Both $F*$ and $G*\urcorner$. Similarly for the other connectives of the propositional calculus.

The translation of $\exists xF$ is $\ulcorner$There is an object that_x is such that $F*\urcorner$.

To obtain the translation of $\exists XF$: Let H be the result of substituting an occurrence of $-x = x$ for each occurrence of Xx in F and let $H*$ be the translation of H. (H has the same number of quantifiers as F.) Then the translation of $\exists XF$ is $\ulcorner$Either $H*$ or there are some objects that_X are such that $F*\urcorner$.

(Since $\ulcorner$There are some objects that_X are such that $F*\urcorner$ properly translates not $\exists XF$, but $\exists X(\exists xXx \mathbin{\&} F)$, we need to disjoin a translation of H, which is equivalent to $\exists X(-\exists xXx \mathbin{\&} F)$, with $\ulcorner$There are some objects that_X are such that $F*\urcorner$ to obtain a translation of $\exists XF$.)

[10] This scheme was given in my 'To Be Is To Be a Value of a Variable (or To Be Some Values of Some Variables)', *The Journal of Philosophy*, 1984, pp. 430–49.

When we apply this translation scheme to the notorious $\exists X - \exists x - (Xx \leftrightarrow -xfx)$, with the predicate letter f given the reading: 'is a member of', we obtain a long sentence that simplifies to 'if some object is not a member of itself, then there are some objects (that are) such that each object is one of them iff it is not a member of itself', a trivial truth.

More generally, the translation of $\exists X \forall x(Xx \leftrightarrow A(x))$ will, as desired, be a sentence that can be simplified to one that is of the form: either there is no object such that . . . it . . . or there are some objects such that an arbitrary object is one of them iff . . . it . . . . And of course, our translation scheme respects the other rules of logic in the sense that if H follows from F and G by one of these rules, and the translations $F*$ and $G*$ of (the universal closures of) F and G are true, then the translation $H*$ of (the universal closure of) H is also true. Our scheme, therefore, respects Frege's rule of substitution of formulae for relation letters as well.

Thus there is a way of interpreting the formulae of the *Begriffsschrift* that is faithful to the usual meanings of the logical operators and on which each comprehension axiom turns out to say something that can also be expressed by a sentence of the form 'if there is something . . . , then there are some things such that anything . . . is one of them and any one of them is something . . .'. Each sentence of this form, it seems fair to say, expresses a *logical* truth if any sentence of English does. It would, of course, be folly to offer a definition of logical truth—as Jerry Fodor once said, failing to take his own advice, 'Never give necessary and sufficient conditions for *anything*'—but I think one would be hard pressed to differentiate 'if there is a rock, then there are some things such that any rock is one of them and any one of them is a rock' from 'if there is a rock then there is something such that if it is not a rock, then it is a rock' on the ground that the former but not the latter expresses a logical truth or on the ground that an intuition is required to see the truth of the former but not the latter.

Three final remarks about definite plural descriptions:

Valid inferences using the construction 'there are some things such that . . . they . . .' that cannot be represented in first-order logic are not hard to come by. The interplay between this construction and definite plural descriptions is well illustrated by the inference

> Every parent of someone blue is red.
> Every parent of someone red is blue.
> Yolanda is red.
> Xavier is not red.
> It is not the case that there are some persons such that
> Yolanda is one of them,
> Xavier is not one of them, and
> every parent of any one of them is also one of them.
> Therefore, Xavier is a parent of someone red.

To see that this is valid, note that it follows from the premisses and denial of the conclusion that Yolanda is either red or a parent of someone red, that Xavier is not, and that every parent of anyone who is red or a parent of someone red is also red or a parent of someone red. Thus there are some people, viz. the persons who are either red or a parent of someone red, such that Yolanda is one of them, Xavier is not one of them, and every parent of any one of them is also one of them, which contradicts the last premiss. This inference may be represented in second-order logic:

$$\forall w \forall z(Bz \,\&\, wPz \rightarrow Rw)$$
$$\forall w \forall z(Rz \,\&\, wPz \rightarrow Bw)$$
$$Ry$$
$$-Rx$$
$$-\exists X(\exists z Xz \,\&\, Xy \,\&\, -Xx \,\&\, \forall w \forall z(Xz \,\&\, wPz \rightarrow Xw))$$
$$\text{Therefore, } \exists z(xPz \,\&\, Rz).$$

In deducing the conclusion from the premisses in the *Begriffsschrift*, one would, of course, substitute $\{a : Ra \vee Ez(aPz \,\&\, Rz)\}$ for the second-order variable X, thus making a move similar to those we have seen Frege make.

It appears that not much in general can be said about 'atomic' sentences that contain definite plural descriptions but do not express statements of identity. 'The rocks rained down', for example, does not mean 'Each of the rocks rained down'. However, if the rocks rained down and the rocks under discussion are the items in pile x, then the items in pile x certainly rained down. If we have learned anything at all in philosophy, it is that it is almost certainly a waste of time to seek an analysis of 'The rocks rained down' that reduces it to a first-order quantification over the rocks in question. It is highly probable that an adequate semantics for sentences like 'They rained down' or 'the sets possessing a rank exhaust the universe' would have to take as primitive a new sort of predication in which, for example 'rained down' would be predicated not of particular objects such as this rock or that one, but rather of these rocks or those. Thus it would appear hopeless to try to say anything more about the meaning of a sentence of the form 'The *K*s *M*' other than that it means that there are some things that are such that they are the *K*s and they *M*. The predication 'they *M*' is probably completely intractable.

About statements of identity, though, something useful if somewhat obvious can be said: 'The *K*s are the *L*s' is true if and only if there is at least one *K*, there is at least one *L*, and every *K* is an *L* and vice versa: $\exists x Kx \,\&\, \exists x Lx \,\&\, \forall x(Kx \leftrightarrow Lx)$. 'They are the *K*s' can also be naturally rendered with the aid of a free second-order variable X: $\exists x Xx \,\&\, \forall x(Xx \leftrightarrow Kx)$. And of course if some things are the *K*s and are also the *L*s, then the *K*s are the *L*s. Frege was not far wrong when he laid down Basic Law (*V*). Of course, from time to time, there will be no set of (all) the *K*s, as the sad history of Basic Law (*V*) makes plain. We cannot always pass from a predicate to an extension of the predicate, a set of things satisfying the predicate. We can, however, always pass to the things satisfying the predicate (if there is at least one), and therefore we cannot always pass from the things to a set of them.

APPENDIX: Proof of 133

Definitions:

Her(F)	$\forall d \forall a(Fd \,\&\, dfa \rightarrow Fa)$	(69 in *Begriffsschrift*)
In(x, F)	$\forall a(xfa \rightarrow Fa)$	
$xf{*}y$	$\forall F(\text{Her}(F) \,\&\, \text{In}(x, F) \rightarrow Fy)$	(76)
FN	$\forall d \forall e \forall a(dfe \,\&\, dfa \rightarrow a = e)$	(115)

The second main theorem of the *Begriffsschrift* (133): $FN \,\&\, xf{*}m \,\&\, xf{*}y \rightarrow [yf{*}m \vee y = m \vee mf{*}y]$.
Proof after four lemmas.

Lemma 1: $bfa \rightarrow bf{*}a$. (91)
Proof: Suppose bfa. Assume Her(F), In(b, F); show Fa. Since bfa and In(b, F), done.

Lemma 2: $cf{*}d \,\&\, df{*}a \rightarrow cf{*}a$. (98)
Proof: Suppose $cf{*}d$ and $df{*}a$. Assume Her(F) and In(c, F); show Fa. Since $cf{*}d$, Her(F), and In(c, F), Fd. If dfb, then since Her(F), Fb; thus In(d, F). Since Her(F) and $df{*}a$, Fa.

Lemma 3: $[c = d \vee cf{*}d] \,\&\, dfa \rightarrow [c = a \vee cf{*}a]$. (108).
Proof: Suppose $[c = d \vee cf{*}d]$ and dfa. If $c = d$, then cfa, whence $cf{*}a$ by lemma 1; if $cf{*}d$, then since dfa, $df{*}a$ by lemma 1, and by lemma 2, $cf{*}a$ again. In any event, $c = a \vee cf{*}a$.

Lemma 4: $FN \,\&\, cfb \,\&\, cf{*}m \rightarrow [b = m \vee bf{*}m]$. (124)
Proof: Suppose FN and cfb. Let $F = \{z : b = z \vee bf{*}z\}$. Suppose $[b = d \vee bf{*}d]$ and dfa. By lemma 3, $[b = a \vee bf{*}a]$. Thus Her(F). If cfa, then by FN, $b = a$, whence $b = a \vee bf{*}a$; thus In(c, F). Therefore if $cf{*}m$, Fm, i.e. $b = m \vee bf{*}m$.

Proof of Frege's theorem: Suppose FN. Let $F = \{z : zf{*}m \vee z = m \vee mf{*}z\}$. Suppose $[df{*}m \vee d = m \vee mf{*}d]$ and dfa. If $df{*}m$, then by lemma 4, $[a = m \vee af{*}m]$, whence $[af{*}m \vee a = m \vee mf{*}a]$; and if $d = m \vee mf{*}d$, then $m = d \vee mf{*}d$, and by lemma 3, $m = a \vee mf{*}a$, whence again $[af{*}m \vee a = m \vee mf{*}a]$. Thus Her($F$). Now suppose $xf{*}m$. Assume xfa. By lemma 4, $[a = m \vee af{*}m]$, whence $[af{*}m \vee a = m \vee mf{*}a]$. Thus In($x, F$). At last, suppose $xf{*}y$. Then Fy, i.e. $yf{*}m \vee y = m \vee mf{*}y$.

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